

## Stability of the Radiative Gradient

We found that if radiative equilibrium obtains, *i.e.*, if the energy is transported through the layers of a star by radiation alone, the temperature gradient is given to good approximation in the stellar interior in terms of the local values of opacity  $\kappa$  (Rosseland mean mass absorption coefficient, see Chap. 8), density  $\rho$ , "interior luminosity"  $L(r)$  (net outward rate of flow of energy through a sphere of radius  $r$ ), and  $r$  by the relation

$$\frac{d}{dr}((1/3)aT^4) = -\frac{\kappa\rho}{c} \frac{L(r)}{4\pi r^2}, \quad (13.1a)$$

which may also be written as

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa\rho}{T^3} \frac{L(r)}{4\pi r^2}. \quad (13.1b)^*$$

We now wish to decide whether the stellar material would be, dynamically, in stable or unstable equilibrium under such a gradient as that given by (13.1). In other words, with such a gradient, would the matter be stable or unstable to small local perturbations?

Suppose the *actual* temperature gradient at the point under consideration is that appropriate to radiative transfer, or, in other words, is given by (13.1). Suppose now that an element of mass  $\delta m$  at the radial distance  $r$  suddenly undergoes an arbitrarily small increase in temperature  $\Delta T(r) > 0$ , where

$$\Delta T(r) \equiv T_{\delta m}(r) - T(r), \quad (13.2)$$

$T_{\delta m}(r)$  being the temperature of the element and  $T(r)$  being the temperature of the unperturbed surrounding material at the point  $r$ . The pressure

\* With a slight generalization of the definition of  $\kappa$ , (13.1a) and (13.1b) apply also when some (or all) of the energy is being transported by *conduction*; see Sect. 16.7.

within the element will increase to a value slightly greater than the surrounding external pressure, and the volume of the element will increase quickly until the internal and external pressures are equal. Thus the density  $\rho_{\delta m}(r)$  of the element will have decreased below the density  $\rho(r)$  of the unperturbed surroundings, so that the excess density

$$\Delta\rho(r) \equiv \rho_{\delta m}(r) - \rho(r) \quad (13.2')$$

of the element will now be negative. The element will accordingly begin to move outward under the influence of the buoyant forces of the surrounding material. We assume that the pressure within the element acting upon the surrounding material is always equal to the external pressure of the surrounding material acting on the element; *i.e.*, we assume that pressure equilibrium obtains at all times:

$$P_{\delta m}(r) = P(r),$$

whence

$$\left(\frac{dP}{dr}\right)_{\delta m} = \frac{dP}{dr}, \quad (13.3)$$

where  $P(r)$  and  $dP/dr$  are the pressure and pressure gradient of the surrounding material. In other words, we are assuming that the time required for pressure equilibrium to become established in the element is small compared with the times of interest. (This assumption is discussed further in Sect. 14.3.)

The values of  $T_{\delta m}(r)$  and  $\rho_{\delta m}(r)$  of the element as it moves outward will depend, in general, on  $P(r)$  (since the pressure forces may do work on the element) and on the way in which the element exchanges heat with its surroundings. In the special case of *adiabatic* motion (no net gains or losses of heat by the element)  $T_{\delta m}(r)$  and  $\rho_{\delta m}(r)$  would be determined only by the value of  $P(r)$  at each point through the adiabatic relations between  $T$  and  $P$  and between  $\rho$  and  $P$ . However, we consider here the general (not necessarily adiabatic) case.

It is clear, now, that for a condition of *stable* equilibrium to exist the density  $\rho_{\delta m}(r)$  of the element must eventually become equal to the unperturbed density  $\rho(r)$  of the surroundings at some point further out; for, when these two densities are equal, the upward buoyant force is exactly balanced by the downward gravitational force, and the net force on the element is zero. In other words, the  $\rho_{\delta m}(r)$  curve shown in Fig. 13.1 must cross the unperturbed  $\rho(r)$  curve at some value of  $r$  greater than that at which the temperature increase occurred. Since we have chosen  $\Delta T$ , and hence  $\Delta\rho$ , to be arbitrarily

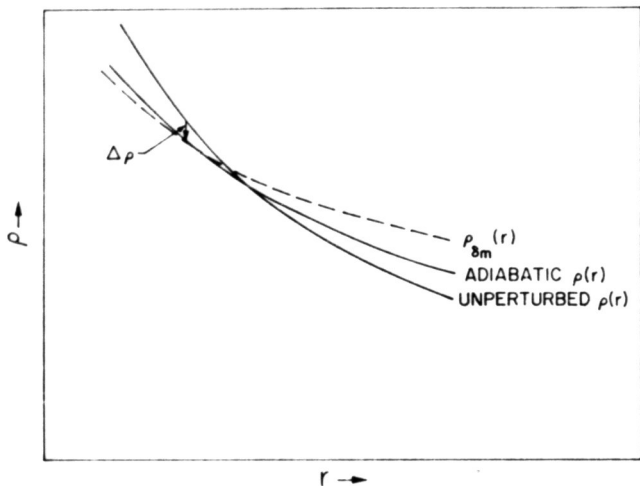


Fig. 13.1 A case of *stability* against convection.  $\Delta\rho$  is to be regarded as an infinitesimal perturbation in density.

small, and since  $\Delta\rho$  is negative in the present example where  $\Delta T$  is positive, then, clearly, a necessary and sufficient general condition for stability of the radiative gradient is that

$$d(\Delta\rho)/dr > 0 \quad (13.4)$$

or

$$(d\rho/dr)_{\delta m} > (d\rho/dr)_{\text{unpert}}, \quad (13.5)$$

where the subscript "unpert" refers to the unperturbed surroundings.

The forms (13.4) and (13.5) of the condition for stability against convection may appear somewhat unconventional, as this condition is usually expressed (see Sect. 13.1) in terms of  $\Delta T$  rather than of  $\Delta\rho$ . However, (13.4) and (13.5) are perfectly general, whereas the usual expressions ((13.6) and (13.7) below) are valid only in the case of uniform chemical composition and under certain assumed conditions (see Sect. 13.1).

Rather than proceeding immediately from the general equations (13.4) and (13.5), we consider, first, in Sect. 13.1 the somewhat more restricted case of uniform chemical composition. This is the case of interest in most applications; moreover, most of the conventional terminology and notation regarding convective stability is based on this case. The more general case of a non-uniform (but continuously varying in space) chemical composition

is considered in Sect. 13.3. (The case of a *discontinuously* varying composition is considered in Sect. 23.6a.) A crude order-of-magnitude estimate of the degree of "superadiabaticity" of the temperature gradient in a convective zone in the deep stellar interior is presented in Sect. 13.2, and a general discussion of convective stability in stars is given in Sect. 13.4.

(Recently, a rigorous study of the conditions for instability against convection, based on (essentially) a detailed linear stability analysis, has been published by Lebovitz [Le65]. This analysis yields the same criterion for convective instability (sometimes called the "Schwarzschild" criterion) as does the conventional, more intuitive, treatment presented in Sect. 13.1. The validity of the Schwarzschild criterion in the case of general relativistic fluid dynamics has been established by Chandrasekhar [Ch65] and by Thorne [Th66a].)

### 13.1 Case of Uniform Chemical Composition

In this section we assume that the chemical composition is constant in space. Hence, in the case of complete (or zero) ionization or dissociation the mean molecular weight  $\mu$  (*cf.* Chap. 15) is constant in space and the "material" pressure  $P$  may be regarded, through the pressure equation of state, as (in

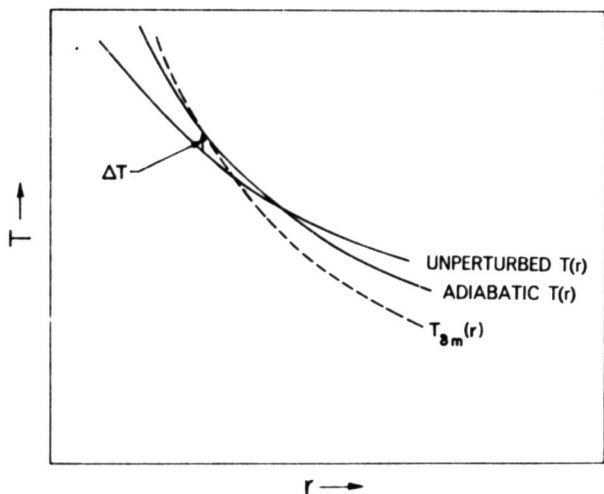


Fig. 13.2 A case of *stability* against convection.  $\Delta T$  is to be regarded as an infinitesimal perturbation in temperature.

general) a function only of density  $\rho$  and temperature  $T$  (i.e., only changes in  $\rho$  and  $T$  contribute to changes in  $P$ ). In the case of partial ionization or dissociation we assume instantaneous chemical equilibrium (see Sect. 9.12), so that, again,  $P$  may be regarded, in general, as a function only of  $\rho$  and  $T$ .

Consider now the rising element of mass  $\delta m$  which was conceptually followed in the introduction to this chapter. The assumption of continuous pressure equilibrium with its surroundings, together with the assumptions stated in the preceding paragraph, require that, when the element attains the same density as its surroundings, its temperature is also the same as that of its surroundings. Hence, in this case of uniform chemical composition we may base the discussion of the criterion for stability against convection on the temperature excess  $\Delta T(r)$ , rather than on the density excess  $\Delta\rho(r)$ , and we shall do so in the remainder of this section.

Reference to Fig. 13.2 and to the discussion in the introduction to this chapter shows that in this case the condition for stability of the radiative gradient is that

$$d(\Delta T)/dr < 0 \quad (13.6)$$

or

$$(dT/dr)_{\delta m} < (dT/dr)_{\text{unpert}}. \quad (13.7)$$

Since  $dT/dr$  is always negative in the stellar interior (except possibly in regions where neutrino energy losses are important, cf. Sects. 17.20, 26.4g, and 26.4h; and possibly in degenerate stellar cores, cf. Eggleton [Eg66]) and since  $T$  is always positive, we may write the necessary and sufficient condition for stability of the radiative gradient (13.7) in the present case in the form

$$\left| \frac{1}{T} \left( \frac{dT}{dr} \right)_{\delta m} \right| > \left| \frac{1}{T} \left( \frac{dT}{dr} \right)_{\text{unpert}} \right|. \quad (13.8)$$

According to (13.8), then, for stability of the radiative gradient the  $T_{\delta m}(r)$  curve must be *steeper* than the unperturbed  $T(r)$  curve of the surroundings.

The condition (13.8) is quite general (within the assumptions stated in the first paragraph of this section) but not very useful for practical computations of stellar models: in order to evaluate  $(dT/dr)_{\delta m}$ , consideration of heat exchange mechanisms with the surroundings of the element would be required and, in general, an iterative procedure of constructing a stellar model might be necessary in this general case. In the important special case where *no energy sources are effective* in the element,\* however, (13.8) can be replaced by a simpler condition. If there are no effective energy sources in

\* For further clarification and discussion of this point, see Chap. 14.

the rising element, then the element can only *lose* heat to its surroundings, either by radiation or by conduction.\* Hence  $T_{\delta m}(r)$  will fall more sharply with increasing  $r$  than would be the case if the element were moving *adiabatically* (*i.e.*, exchanging no heat, in the net, with its surroundings); thus the  $T_{\delta m}(r)$  curve will in this case be *steeper* than the adiabatic  $T(r)$  curve:

$$\left| \left( \frac{dT}{dr} \right)_{\delta m} \right| \geq \left| \left( \frac{dT}{dr} \right)_{\text{ad}} \right|. \quad (13.9)$$

It is clear, then, that whenever (13.9) is valid, the necessary and sufficient condition (13.8) can be replaced by the simpler sufficient condition for stability of the radiative gradient:

$$\left| \frac{1}{T} \left( \frac{dT}{dr} \right)_{\text{ad}} \right| > \left| \frac{1}{T} \left( \frac{dT}{dr} \right)_{\text{unpert}} \right|, \quad (13.10)$$

since satisfaction of (13.10) will guarantee satisfaction of (13.8), provided that (13.9) is valid.

In case (13.9) is not satisfied, as may be the case if effective energy sources (such as nuclear sources) are present in the element, then of course (13.10) is not strictly the correct condition for stability and the more general condition (13.8) should be used. This case, however, presents no difficulty in practice since, as will be shown in Sect. 13.2,  $(dT/dr)_{\delta m}$  is likely to be equal to  $(dT/dr)_{\text{ad}}$  to high accuracy even in those parts of a star (generally, the deeper regions) where nuclear energy sources *are* effective. Consequently, the condition (13.10) may be used, in practical calculations, as the condition for stability even in this case, although (13.8) is really the rigorously correct condition under the assumed conditions. The condition (13.10) is universally used in calculations of chemically homogeneous stellar models.

If (13.8) is satisfied, the inertia of the upward-moving element will cause it to overshoot its equilibrium position (defined by the intersection of the  $T_{\delta m}(r)$  and the  $T(r)$  curves). Subsequently, however, a restoring force on the element will develop because  $T_{\delta m}(r)$  will then be less than  $T(r)$ , whence  $\rho_{\delta m}(r) > \rho(r)$ , and the gravitational force on the element will exceed the buoyant force. The motion of the element will thus eventually be reversed, and there is clearly no tendency for convective motions to develop, *i.e.*, the material is stable against convection.

We now assume that the radiative gradient is stable, *i.e.*, that (13.8) is satisfied at the point under consideration. For simplicity, we shall assume that the simpler condition (13.10) can be used as the stability criterion. To

\* As is pointed out in Chap. 14, the heat loss here refers only to the "horizontal" loss, and has nothing to do with the heat gains or losses associated with *thermal equilibrium* (*cf.* Chap. 5).

describe the *unperturbed* temperature gradient at the point of interest, we may make use of the effective polytropic index which was defined in Chap. 12 by the relation

$$(d \ln P/d \ln T)_r - 1 = n_e(r),$$

which may be written in the form

$$\left(\frac{1}{T} \frac{dT}{dr}\right)_{\text{unpert}} = \frac{1}{n_e + 1} \left(\frac{1}{P} \frac{dP}{dr}\right). \quad (13.11)$$

Equation (13.11) expresses the actual, unperturbed temperature gradient at the point of interest in terms of the pressure gradient and the effective polytropic index at that point.

For the *adiabatic* relation between  $P$  and  $T$  at the point of interest, we have (cf. (9.88))  $(\Gamma_2 - 1)/\Gamma_2 \equiv (d \ln T/d \ln P)_{\text{ad}}$ , which may also be written in the form

$$\left(\frac{1}{T} \frac{dT}{dr}\right)_{\text{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2} \left(\frac{1}{P} \frac{dP}{dr}\right). \quad (13.12)$$

Using (13.11) and (13.12) in (13.10), we obtain

$$n_e > \frac{1}{\Gamma_2 - 1} \quad (13.13)$$

as the condition expressing stability of the radiative gradient under the assumed conditions when the simple condition (13.10) is appropriate. (When the more general condition (13.8) must be used, the adiabatic exponent  $\Gamma_2$  in (13.13) may be replaced by the polytropic exponent  $\Gamma_2'$  which relates  $P$  and  $T$  during the motion of the moving element. However, in this book we shall always assume that the simpler condition (13.10) is an adequate criterion for stability of the radiative gradient under the assumed conditions.)

For example, if the equation of state is of the perfect gas law form,  $P = \text{const. } \rho T$ , we have  $\Gamma_2 = \gamma = c_p/c_v$  (cf. Sect. 9.14b), and (13.13) becomes  $n_e > 1/(\gamma - 1)$ . For a non-relativistic, perfect monatomic gas  $\gamma = 5/3$  (cf. Sect. 10.7a), whence  $n_e > 1.5$  for stability of the radiative gradient. As another example, consider the case where the gas pressure is negligible compared with the radiation pressure. Then we have  $\Gamma_2 = 4/3$  (cf. Sect. 9.16), so that  $n_e > 3$  for stability against convection. In an actual star  $n_e$  usually decreases inward, at least at points below the regions of hydrogen and helium ionization but not too near the stellar center, cf. Part II of this book. Consequently, it follows that radiation pressure may be expected to increase the value of  $r$  below which instability against convection exists over the value which would

obtain in the absence of radiation pressure (this expectation is confirmed in the models of Deinzer and Salpeter [De64] and Meggitt [Me65]). Thus the effect of radiation pressure is to reduce the steepness of the adiabatic temperature gradient at a given point and thus to favor instability against convection.

To summarize: At every point in a chemically homogeneous star at which the radiative gradient is stable, we must have, combining (13.10) and (13.12),

$$\left(\frac{d \ln T}{d \ln P}\right)_r < \frac{\Gamma_2 - 1}{\Gamma_2}. \quad (13.14)$$

We now consider the case in which the radiative gradient is *unstable*, i.e., (13.8) is not satisfied. In this case the radiative temperature gradient that would prevail under perfectly static conditions (i.e., with all existing convective motions artificially suppressed) is *steeper* than the gradient  $(dT/dr)_{\text{ad}}$  which a rising element would follow. This clearly represents a case of unstable

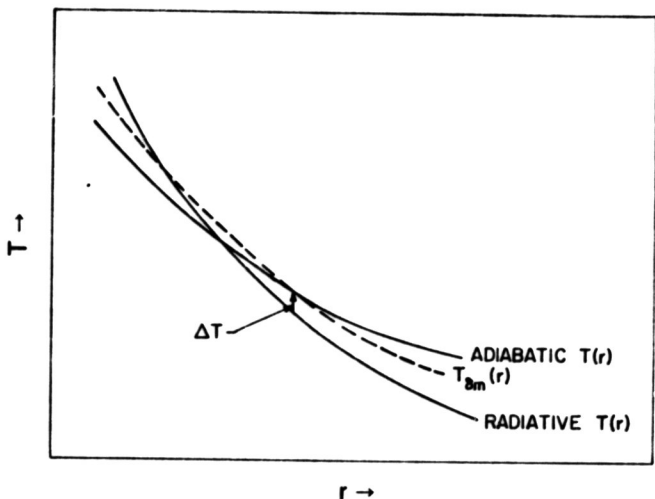


Fig. 13.3 A case of *instability* against convection.  $\Delta T$  is to be regarded as an infinitesimal perturbation in temperature.

equilibrium; for, if the temperature of a small element of matter suddenly increased by an infinitesimal amount, the element would rise, following the  $T_{adm}(r)$  curve in Fig. 13.3, and would continue to rise until it reached a point, considerably further out, at which the  $T_{adm}(r)$  curve and the radiative  $T(r)$



curve crossed again, or else until the element had dissolved through turbulent mixing. Similarly, a slight decrease in the temperature of the element would cause it to descend toward the center of the star. If no effective energy sources are present in the element, then  $(dT/dr)_{\delta m}$  will clearly be intermediate in value between  $(dT/dr)$ , the "static" gradient, and  $(dT/dr)_{ad}$ , the adiabatic gradient. In this case, then, the actual average gradient in the convection zone will also be steeper than  $(dT/dr)_{ad}$  under conditions of instability against convection.

Under these conditions, then, upward and downward convection currents would be set up which would carry large amounts of energy from the interior of the star, thus effecting an overall decrease in the steepness of the  $T(r)$  curve which would otherwise obtain in a perfectly static condition. Eventually a "steady" situation would be realized, in which both radiation and convection compete as transport mechanisms, each carrying a part of the total energy flux. The actual  $T(r)$  curve within the convective region would then have a slope intermediate between that of the radiative  $T(r)$  curve and that of the adiabatic  $T(r)$  curve. That is, the actual  $T(r)$  curve would be *superadiabatic* (steeper than the adiabatic  $T(r)$  curve) but *less steep* than the radiative  $T(r)$  curve. It is clear that the actual  $T(r)$  curve in the convecting region must be superadiabatic if there are no effective energy sources in the region; if the actual gradient were exactly adiabatic under these conditions, then there would be no energy transport by convection, and hence no "driving force" for the convection. The extent to which the superadiabatic gradient would differ from the adiabatic would depend on the ratio of the *excess* heat energy that each unit mass of the gas would have to carry away from the interior to the total internal thermal energy of the unit mass.

### 13.2 Estimate of the Degree of Superadiabaticity in the Deep Interior

In the deep interior of a star the superadiabatic gradient is always only negligibly steeper than the adiabatic one. This question of the magnitude of the superadiabaticity will be discussed much more fully in Chap. 14; in this section, however, we shall give a rough order-of-magnitude estimate of the extent to which the superadiabatic gradient exceeds the adiabatic in the deep interior.

Consider a spherical shell of thickness  $A$  within a convective region in the deep interior of a star (see Fig. 13.4). We assume that there is no nuclear energy production occurring in the regions of interest. We take  $A$  to be the "mixing length" (cf. Chap. 14); *i.e.*, the "mean free path" of an average

turbulence element. Also, let  $t$  be the "mean life" of an average turbulence element, or the time required for the element to convect through the distance  $A$ . It is clear, then, that a time of the order of  $t$  must elapse before an amount of energy  $L(r)t$  incident on the lower boundary of the shell in time  $t$  can emerge from the top of the shell at radius  $r + A$ , where  $L(r)$  is the net rate at which energy is carried by convection outward through a sphere of radius  $r$  (we are here neglecting the energy transport by radiation, since the present

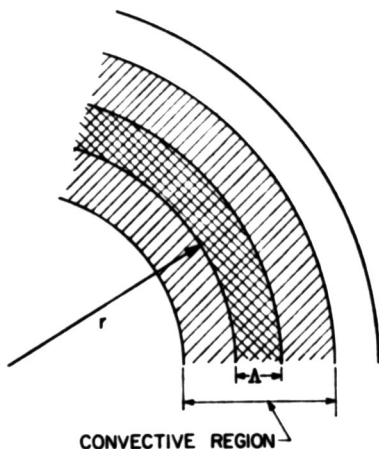


Fig. 13.4 Illustration for estimating the superadiabaticity of the temperature gradient.

argument is only an order-of-magnitude one). (Note that, if we consider luminosities averaged over times comparable to or longer than  $t$ , we must have  $L(r) = L(r + A)$  if there is negligible gravitational or nuclear energy released within the shell of thickness  $A$  in the time  $t$ .) The matter of mass  $\Delta M$  within the shell effectively "stores" this energy  $L(r)t$  incident in time  $t$  on the bottom of the shell, and then releases it at the top of the shell. Thus, in time  $t$  each unit mass of material in the shell will absorb the amount of energy  $L(r)t/\Delta M$  and will consequently "heat up" by the amount  $\Delta T = L(r)t/\Delta M/c_V$ , where we take  $c_V \approx (3/2)\mathcal{R}/\mu$  as the specific heat per unit mass at constant volume. If the average temperature of the material during the time  $t$  is  $T$ , then the excess of the superadiabatic gradient over the adiabatic will be of order

$$\frac{\Delta|(dT/dr)|}{|(dT/dr)_{ad}|} \sim \frac{\Delta T}{T} \sim \frac{L(r)t/\Delta M}{c_V T}. \quad (13.15)$$

For typical conditions in the deep interior of a star of approximately solar type we may take  $L(r) \sim 10^{33}$  erg/sec,  $t \sim 10^6$  sec (see Chap. 14\*),  $\Delta M \sim (1/10)M_{\odot} \sim 10^{32}$  gm, and  $T \sim 10^7$ °K. Thus  $L(r)t/\Delta M \sim 10^7$  erg/gm is the amount of energy absorbed per gram during the time  $t$ , and  $c_V T = (3/2)(\mathcal{R}/\mu)T \sim 10^8 \times 10^7 \sim 10^{15}$  erg/gm is the average internal energy per gram of the stellar material. Hence the fractional excess of the actual temperature gradient in the convective region over the adiabatic is

$$\frac{\Delta|dT/dr|}{|(dT/dr)_{ad}|} \sim 10^{-8},$$

which is a very small excess indeed.

Thus, in a convective region in the deep interior of a star we expect that the actual, superadiabatic temperature gradient can be approximated to a very high degree of accuracy by the *adiabatic* temperature gradient. Hence, *within* a convective region in the deep interior the temperature gradient should be given very accurately by the relation

$$\frac{1}{T} \frac{dT}{dr} = \frac{\Gamma_2 - 1}{\Gamma_2} \frac{1}{P} \frac{dP}{dr}. \quad (13.16)$$

*Outside* such a region, where the material is in radiative equilibrium, the temperature gradient is given by (13.1):

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa \rho}{T^3} \frac{L(r)}{4\pi r^2}. \quad (13.17)$$

It should be mentioned that in the outer stellar layers, where the internal energy per unit mass is small due to the relatively low temperatures, the actual temperature gradient in a convective region may depart appreciably from the adiabatic gradient. Hence, in the outer layers a slightly superadiabatic gradient may not provide an adequate representation of the actual temperature gradient (see Chap. 14).

### 13.3 Case of Non-Uniform Chemical Composition

We consider now the more general case where the chemical composition varies (continuously) with radial distance. The "material" pressure  $P$  must now be considered a function not only of  $\rho$  and  $T$ , but also of mean molecular

\* At the end of Sect. 14.6 an order-of-magnitude derivation of the value of  $\Delta T/T$ , as well as values of other convective quantities, in the deep stellar interior will be presented, which does not require a priori knowledge of the value of  $t$ .

weight  $\mu$ . In this case it is necessary to go back to the general forms (13.4) and (13.5) of the condition for stability against convection. We use  $\ln P$  instead of  $r$  as the independent radial variable, where  $P$  is the total pressure (which always increases inward in a star in hydrostatic equilibrium). Equation (13.4) becomes

$$\frac{d(\Delta\rho)}{d \ln P} < 0 \quad (13.18)$$

and (13.5) becomes

$$\left(\frac{d \ln \rho}{d \ln P}\right)_{\delta m} < \left(\frac{d \ln \rho}{d \ln P}\right)_{\text{unpert}} \quad (13.19)$$

Before expressing (13.19) in terms of temperature gradients, let us digress for a moment to consider from a physical standpoint the fate of an element of mass  $\delta m$  whose temperature  $T_{\delta m}(P)$  abruptly increases over the temperature  $T(P)$  of its surroundings by a small amount and whose density  $\rho_{\delta m}(P)$  decreases (still assuming pressure equilibrium of the element with its surroundings) below the density  $\rho(P)$  of its surroundings by a small amount. Just as in Sect. 13.1, the element will move outward because of the presence of the unbalanced buoyant forces. As the element moves outward, we may assume that the mean molecular weight in the element remains constant and equal to the value characteristic of the location in the star where the element originated.\* Because of the assumed spatial variation of mean molecular weight, a difference between the mean molecular weight  $\mu_{\delta m}$  of the element and the mean molecular weight  $\mu(P)$  of its immediate surroundings develops. In the case of stability against convection, the element will eventually reach a position where  $\rho_{\delta m}(P) = \rho(P)$  and the buoyancy and the gravitational forces balance each other. After a slight overshoot and possibly some oscillations about this position, the element will soon (at least for the moment) come to rest. However, because  $\mu_{\delta m} \neq \mu(P)$  at this "equilibrium" position and because the "material" pressure  $P$  is, through the pressure equation of state, a function of  $\rho$ ,  $T$ , and  $\mu$ , it follows that  $T_{\delta m}(P) \neq T(P)$ , *i.e.*, although the element experiences zero net force and is (temporarily) at rest, it has a different temperature from that of its immediate surroundings.

The subsequent fate of the element depends on the value of the time, say  $t_d$ , during which it can retain its identity against diffusive mixing with its

\* We ignore here and until the end of this section any possible changes in  $\mu$  resulting from ionization and/or dissociation within the element; see the last paragraph in this section.

surroundings, as compared with the value of the time, say  $t_h$ , required for significant heat exchanges with its surroundings. If  $t_d \ll t_h$ , the element will mix with its surroundings and lose its identity. If, on the other hand,  $t_d \gg t_h$ , the element will exchange heat with its surroundings and tend to acquire their temperature, with a consequent change in density from that of its surroundings. This change in density will lead to further motions of the element, either inward or outward, but on a much longer time scale than that associated with the original motion of the element. If  $\mu(P)$  decreases outward (as is normally the case with evolving stars, see Sects. 23.6 and 26.4) and if the equation of state is the perfect gas law (possibly modified to include black body radiation pressure), then when the element has just come to rest, we will have  $T_{dm}(P) > T(P)$ , and the element will cool down, contract, and so move inward. It is clear that the element, if it did not first dissolve through diffusive mixing with its surroundings, would eventually have to return to the level from which it originated. Hence, in the present case of  $\mu$  decreasing outward, satisfaction of the condition (13.19) will insure stability not only against ordinary convection, but also against slow, "convective"-like motions having a much longer time scale than for ordinary convective motions. If  $\mu(P)$  increases outward, then satisfaction of the condition (13.19) will, again, insure stability against ordinary convection. However, it is easy to see that the material will in this case be *unstable* against slow, "convective"-like mixing on a much longer time scale than for ordinary convection; we may refer to such slow, "convective"-like motions as "quasi-convection." This case of  $\mu(P)$  increasing outward, however, does not ordinarily arise in stars during the course of slow, quasi-static evolutionary processes (disregarding effects of ionization and/or dissociation; see the end of this section).

At any rate, (13.19) is certainly the correct criterion for stability against convection as understood in the ordinary sense. We now wish to express this criterion in terms of temperature gradients.

In order to express  $(d \ln \rho / d \ln P)_{dm}$  in terms of a temperature gradient, we recall that we have assumed that the mean molecular weight  $\mu$  of the element does not change during its motion.\* Hence, as we follow the element, the pressure change within the element is made up only of changes in density  $\rho$  and temperature  $T$ , not in  $\mu$ . We write the pressure equation of state in the general form  $P = P(\rho, T, \mu)$  and take the logarithmic differential, keeping  $\mu$  fixed. We obtain

$$d \ln P = \chi_\rho d \ln \rho + \chi_T d \ln T, \quad (13.20)$$

\* See the earlier footnote in this section.

where (cf. (9.81) and (9.82))

$$\chi_\rho \equiv \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_{T, \mu}, \quad (13.21)$$

$$\chi_T \equiv \left( \frac{\partial \ln P}{\partial \ln T} \right)_{\rho, \mu}. \quad (13.22)$$

We then obtain from (13.20)

$$\left( \frac{d \ln \rho}{d \ln P} \right)_{\delta m} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho} \left( \frac{d \ln T}{d \ln P} \right)_{\delta m}, \quad (13.23)$$

where we assume that, in view of the arbitrarily small initial temperature perturbation in the element,  $\chi_\rho$  and  $\chi_T$  have the same values in the element as in the unperturbed surroundings at each level.

In the case of  $(d \ln \rho / d \ln P)_{\text{unpert}}$ , we must take into account the change in  $\mu$  that is experienced as one moves about in the unperturbed surroundings (because of the non-uniform composition). In this case we must have, instead of (13.20),

$$d \ln P = \chi_\rho d \ln \rho + \chi_T d \ln T + \chi_\mu d \ln \mu, \quad (13.24)$$

where

$$\chi_\mu \equiv \left( \frac{\partial \ln P}{\partial \ln \mu} \right)_{\rho, T}, \quad (13.25)$$

so that

$$\left( \frac{d \ln \rho}{d \ln P} \right)_{\text{unpert}} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho} \left( \frac{d \ln T}{d \ln P} \right)_{\text{unpert}} - \frac{\chi_\mu}{\chi_\rho} \left( \frac{d \ln \mu}{d \ln P} \right). \quad (13.26)$$

Using (13.23) and (13.26) in (13.19), and replacing the subscripts "unpert" by "rad" (for "radiative"), we obtain as the (necessary and sufficient) condition for stability against convection in regions of continuously varying (in space) composition

$$\left( \frac{d \ln T}{d \ln P} \right)_{\text{rad}} < \left( \frac{d \ln T}{d \ln P} \right)_{\delta m} - \frac{\chi_\mu}{\chi_T} \frac{d \ln \mu}{d \ln P}, \quad (13.27)$$

which reduces to the usual condition (see Sect. 13.1) when  $\mu$  is constant in space.

If there are no (nuclear) energy sources in the regions of interest, then (13.27) may be replaced by a simpler and more useful (but less general) con-

dition. In this case we have (see Sect. 13.1)  $(d \ln T/d \ln P)_{\delta m} \geq (d \ln T/d \ln P)_{\text{ad}}$ , so that (13.27) becomes

$$\left(\frac{d \ln T}{d \ln P}\right)_{\text{rad}} < \left(\frac{d \ln T}{d \ln P}\right)_{\text{ad}} - \frac{\chi_{\mu} d \ln \mu}{\chi_T d \ln P} \quad (13.28a)$$

or, in terms of  $(\Gamma_2 - 1)/\Gamma_2 \equiv (d \ln T/d \ln P)_{\text{ad}}$  (*cf.* Sect. 9.14),

$$\left(\frac{d \ln T}{d \ln P}\right)_{\text{rad}} < \frac{\Gamma_2 - 1}{\Gamma_2} - \frac{\chi_{\mu} d \ln \mu}{\chi_T d \ln P}, \quad (13.28b)$$

which differs from (13.14) only in the presence of the last term. Condition (13.28b) is sufficient (but not necessary) in the same sense as is condition (13.10).

Consider, for example, an equation of state appropriate to a mixture of an ideal gas and black body radiation:

$$P = \frac{\mathcal{R}}{\mu} \rho T + \frac{1}{3} a T^4, \quad (13.29)$$

where all symbols have their usual meaning (see Chap. 1). We readily obtain from (13.29) the results  $\chi_{\mu} = -\beta$  and  $\chi_T = 4 - 3\beta$ , where  $\beta$  is the ratio of gas to total (gas plus radiation) pressure. Hence (13.28b) becomes in this case

$$\left(\frac{d \ln T}{d \ln P}\right)_{\text{rad}} < \frac{\Gamma_2 - 1}{\Gamma_2} + \frac{\beta}{4 - 3\beta} \frac{d \ln \mu}{d \ln P}, \quad (13.30)$$

an equation which was derived by Sakashita, Ono, and Hayashi [Sa59] and which has been used in a number of investigations of stellar evolution (for example, Stothers [St66c, 66d] and Hofmeister, Kippenhahn, and Weigert [Ho64]). According to this equation, a mean molecular weight  $\mu$  which *increases* (continuously) inward (the usual situation in evolving stars, *cf.* Sects. 23.6 and 26.4) tends to *stabilize* the corresponding regions against convection, since  $d \ln \mu/d \ln P > 0$  in this case.

Consider now the case where the chemical composition is constant in space but where  $\mu$  varies with position as a result of ionization and/or dissociation. At least in a static star chemical equilibrium (*cf.* Sects. 9.12 and 13.1) may be assumed to obtain in the unperturbed surroundings, and  $\mu$  for the surroundings is then a function of  $\rho$  and  $T$ . The effects of the variable  $\mu$  are then absorbed into the  $\chi_{\rho}$  and  $\chi_T$  in (13.24). If instantaneous chemical equilibrium is assumed to obtain in the perturbed element, then  $\mu$  for the element is also a function of  $\rho$  and  $T$ , and effects of the variable  $\mu$  do not have to be considered explicitly. Hence, in this case of instantaneous chemical

equilibrium the usual condition (13.10) or (13.28b) (without the last term) for stability against convection obtains, even in regions where spatial variations in  $\mu$  are brought about by ionization and/or dissociation.

If, on the other hand, instantaneous chemical equilibrium is *not* assumed to obtain in the perturbed element, then a term similar to the last term in (13.24) must be added to the right side of (13.20) for the element. Also, the values of  $\chi_\rho$  and  $\chi_T$  for the element could no longer be assumed to be the same as for the unperturbed surroundings at the given level, even if the initial temperature perturbation of the element were infinitely small. It is clear that a much more complicated expression than (13.28b) for stability against convection would result for this last case, which is not ordinarily of much interest anyway.

### 13.4 General Discussion of Stability Against Convection

In view of the above considerations, we see that, in general, the onset of convection at some point in a star is governed by the steepness of the radiative temperature gradient relative to the adiabatic gradient at that point, corrected, if necessary, for the presence of a non-uniform chemical composition. In the deep interior of a star, where  $\Gamma_2$  may be essentially constant or slowly varying (and greater than 4/3), the steepness of the adiabatic gradient (with respect to pressure) is essentially constant, and therefore the onset of convection will be determined essentially by the steepness of the local *radiative* gradient (with respect to pressure). (We are here and in the next few paragraphs assuming that the correction term for the effects of a non-uniform composition is negligible or zero.) The radiative gradient with respect to pressure may be obtained by combining the equation of radiative transfer,

$$-\frac{1}{T} \frac{dT}{dr} = \frac{3}{4ac} \frac{\kappa \rho}{T^4} \frac{L(r)}{4\pi r^2},$$

and the equation of hydrostatic equilibrium,

$$-\frac{1}{P} \frac{dP}{dr} = G \frac{M(r)}{r^2} \frac{\rho}{P},$$

to obtain

$$\left( \frac{d \ln T}{d \ln P} \right)_{\text{rad}} = \frac{3}{16\pi acG} \frac{\kappa P}{T^4} \cdot \frac{L(r)}{M(r)}. \quad (13.30')$$

Since the ratio  $P/T^4$  is generally (but not always) a slowly varying function of position in a star in radiative equilibrium (this statement will be justified in



Sect. 23.2), it is seen that the onset of convection in the deep interior is determined primarily by the values of  $\kappa$  and  $L(r)/M(r)$ . A large value of  $\kappa$  means that a steeper temperature gradient is required for a given flux of energy to be transported by radiation than a small value. In the deep interior of an actual star  $\kappa$  generally decreases toward the center; this effect then serves to *hinder* the onset of convection. The factor  $L(r)/M(r)$ , however, may tend to *favor* the onset of convection as  $r$  decreases. If the energy sources are strongly concentrated toward the center, then  $L(r)$  will remain equal to  $L$  until  $r$  becomes very small. But the interior mass  $M(r)$  will steadily decrease with decreasing  $r$  (as  $r^3$  at points sufficiently close to the center). Thus, as one descends toward the center, the ratio  $L(r)/M(r)$  will probably become sufficiently large at some point to initiate convection, and a convective core will result. Hence stars with highly concentrated energy sources will almost certainly have convective cores. If the energy sources are more-or-less uniformly distributed, however, then  $L(r)$  may decrease with decreasing  $r$  almost as rapidly as  $M(r)$ , so that the ratio  $L(r)/M(r)$  may increase only very slowly (or perhaps remain constant) with decreasing  $r$ . The radiative gradient may therefore never become steep enough to initiate convection. Stars with not-so-highly concentrated energy sources therefore may or may not have convective cores.

In the outer layers of a star, now, where  $L(r) \simeq L$  and  $M(r) \simeq M$ , the factor  $L(r)/M(r)$  no longer plays any direct role in determining the onset of convection. In these regions, however, the steepness of the *adiabatic* temperature gradient is not constant, but is very sensitive to the state of ionization (or dissociation) of the dominant constituents (such as hydrogen and helium) of the stellar envelope. In a region of partial ionization of hydrogen or helium  $\Gamma_2$  will drop to a value near unity, so that the adiabatic gradient will become relatively small, as may be seen from the relation

$$\frac{\Gamma_2 - 1}{\Gamma_2} = \left( \frac{d \ln T}{d \ln P} \right)_{\text{ad}}$$

(see Sect. 9.18). If the adiabatic gradient becomes flatter than the radiative gradient (as will almost certainly be the case for hydrogen ionization, and probably also for one or both stages of helium ionization), then a convective zone will be initiated, in which convection *may or may not* carry an appreciable fraction of the total flux (see Chap. 14).

The role played by  $\kappa$  in the outer layers is the same as in the deep interior; *i.e.*, an inward *increase* in the value of  $\kappa$  will favor *convective* transfer, and an inward *decrease* in the value of  $\kappa$  will favor *radiative* transfer. Moreover, a small value of  $\Gamma_2$  will also tend to favor convective transfer. It might be added that both of these factors that favor convective transfer are strongly

operative in the region of hydrogen ionization. Therefore, probably all stars that contain a large abundance of hydrogen (which includes practically all stars), and which are not so hot that hydrogen is completely ionized even at the photosphere, have hydrogen convection zones near their surfaces. The importance of such a hydrogen convection zone in influencing the *structure* of the star, however, depends on the fraction of the flux carried by convection and on the depth of the convection zone. These effects will be discussed in later chapters.

As was pointed out in Sect. 13.3, a non-uniform chemical composition for which the mean molecular weight increases inward continuously (*cf.* Sects. 23.6 and 26.4) tends to stabilize the corresponding regions against convection.

Another useful and enlightening viewpoint is obtained by relating the question of convective stability to the question of whether the local specific entropy  $S$  (entropy per unit mass, say) of the unperturbed material is increasing or decreasing inward in the star. This viewpoint is useful at least in the case of uniform chemical composition and in case no irreversible processes (such as nuclear reactions, *cf.* Sects. 3.1, 9.12, and the paragraph following (17.75'')); or viscous heating) are occurring at the point under consideration. We shall see that in these cases the specific entropy in a star in hydrostatic equilibrium always *decreases inward* in convectively *stable* regions.

To show this, we regard total pressure  $P$  and temperature  $T$  as the independent thermodynamic variables, and we consider an infinitesimal *reversible* process, in which  $S$  changes by the amount  $dS$ . Using the relation (valid for reversible processes, *cf.* Chap. 9)  $TdS = dE + Pd(1/\rho)$  ( $E$  = internal energy per unit mass), assuming that  $E$  is a function only of  $\rho$  and  $T$ , and using some of the identities established in Sects. 9.11 and 9.14, we obtain

$$dS = c_p(\nabla - \nabla_{ad})d \ln P, \quad (13.31)$$

where  $c_p$  is the specific heat per unit mass at constant pressure, and where we have used the abbreviations (*cf.* Chap. 14)

$$\nabla \equiv (d \ln T / d \ln P), \quad (13.32)$$

$$\nabla_{ad} \equiv (\Gamma_2 - 1) / \Gamma_2 = (d \ln T / d \ln P)_{ad}. \quad (13.33)$$

In the present application of (13.31)  $\nabla$  may be taken as the actual temperature gradient (with respect to pressure) of the unperturbed material at the point of interest,  $\nabla_{ad}$  as the corresponding adiabatic gradient (*cf.* (13.12)), and  $d \ln P$  as an increment in  $\ln P$  of the unperturbed material corresponding to an inward or outward displacement of the point of observation (see remarks made in connection with (9.9') and (9.9'')). Since

$P$  always increases inward in a star in hydrostatic equilibrium, it follows from (13.31) that  $S$  in such a star always *decreases* inward in convectively *stable* regions ( $\nabla < \nabla_{ad}$ , cf. (13.10)) under the assumed conditions, Q.E.D. In convectively *unstable* regions ( $\nabla > \nabla_{ad}$ ),  $S$  would clearly *increase* inward in a star in hydrostatic equilibrium if convective motions were artificially suppressed. Hence, in the present case of uniform composition where no irreversible processes are occurring and for a star in hydrostatic equilibrium, the local behavior of the specific entropy  $S$  (assuming convective motions to be suppressed) can also be used as a criterion for convective stability: The material is convectively *stable or unstable* according as  $S$  (in the absence of convective motions) *decreases or increases* inward, respectively.

Note that  $S$  is *constant* with depth in the absence of viscous heating in a convective zone in adiabatic equilibrium ( $\nabla = \nabla_{ad}$ ). As we have pointed out earlier in this chapter, the actual gradient in a real convection zone must be slightly *superadiabatic* ( $\nabla$  only slightly greater than  $\nabla_{ad}$ ), if no nuclear reactions are occurring, so that under these conditions  $S$  *increases inward* slowly (perhaps negligibly so, cf. Chap.20) in an actual convection zone.